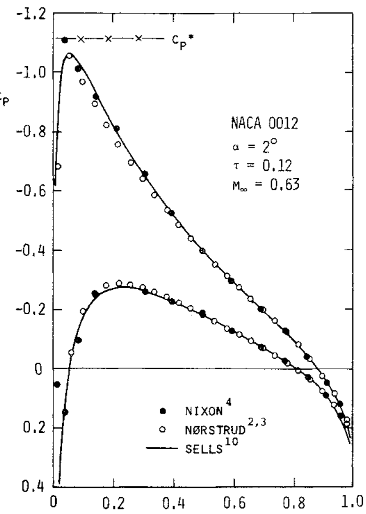

Fig. 4 Pressure distribution on a lifting NACA 0012 airfoil in subcritical flow.

at the points $x = 0.0832$ and 0.9168, the agreement away from these singular points is satisfactory. It should be noted that the maximum value of the pressure coefficient (at $x=0.5$) is -0.2980 for $M_\infty = 0.8085$ (and $\tau=0.06$), whereas $c_p = -0.2902$ for $M_\infty = 0.8$.

The solutions of Nφrstrud as presented in Figs. 1 and 2 are based on some approximation in the description of the flowfield (see Ref. 8 for a more detailed discussion). In particular, the solution[2] of Fig. 1 is obtained by describing the flowfield according to the functional relationship for the velocity decay

$$u(x,[1-M_\infty^2]^{1/2}y) = u(x,0)\exp[-(1-M_\infty^2)^{1/2}y/r(x)] \quad (2)$$

where the function $r=r(x)$, which depicts the lateral extent of the compressibility effects, is a universal function for a given airfoil shape. For a parabolic arc profile one can write $r(x) = |[u_i(x)-u_\infty)/(4\tau u_\infty)]|$ and with Eq. (2) one can evaluate $u(x,y)$ as a function of $u(x,0)$. This has been done in connection with the data given by Magnus et al.[9] and the results are shown as Mach contours in Fig. 3. For values of $|y|<r$ the comparison of field values is good, at least, it shows a qualitative agreement on which some conclusions can be based. Since Nixon in his method[1] uses field points as solution points, it would be of fundamental interest to see a direct comparison of calculated and assumed decay functions.

The last comment to be made is only indirectly related to the work presented in Ref. 1. The present author does agree with Nixon when he states that the approximate evaluation of the field integral in the "standard" integral equation method is adequate for subcritical flows.[1] This is, e.g., shown in Fig. 4 where a comparison is made between the extended[4] and a standard[2,3] integral equation solution together with the "exact" solution[10] obtained by Sells method. However, a conflict occurs when Nixon argues that the method of solution presented in Ref. 3 for lifting flows is incorrect and nonunique[4]. To present a formal disputal is difficult since Nixon[4] starts his discussion with an equation which is not found in Ref. 3. This is further complicated by sign errors in an equation (labeled 5 in Ref. 4) which is derived from this starting equation.

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Reply to Author to H. Nørstrud

D.Nixon*

Queen Mary College, London, England

IN his Comment, Nφrstrud makes three points, namely: 1) that the comparison of the method presented in Ref. 1 with other methods[2-4] is erroneous and misleading; 2) that the statement made on the validity of the approximations used in the 'standard' integral equations method may be misleading; 3) that the argument present in Ref. 5 regarding the formulation of the integral equations for lifting flows is incorrect. These points will be considered in sequence.

The basic differential equation used in Ref. 1 is

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{z}\bar{z}} = \bar{\phi}_{\bar{x}}\bar{\phi}_{\bar{x}\bar{x}} \quad (1)$$

where, if M_∞ is the freestream Mach number, $k\ (M_\infty)$ is a transonic parameter, $\beta = (1-M_\infty^2)^{1/2}$, and γ is the ratio of specific heats,

$$\bar{\phi} = \frac{k(\gamma+1)}{\beta^2}\phi,\ \bar{x}=x,\ \bar{z}=\beta z \quad (2)$$

where $\phi\ (x,z)$ is the perturbation velocity potential and (x,z) is the Cartesian co-ordinate system in real space.

The tangency boundary condition for Eq. (1) is

$$\bar{\phi}_z(\bar{x},+0) = [(\gamma+1)k(M_\infty)/\beta^3]\tau z'_T(x) \quad (3)$$

where τ is the thickness/chord ratio of the aerofoil, and $z_T(x)$ is a shape function for a family of aerofoils.

A solution for the velocity variable $\bar{u}(\bar{x},\bar{z})\ (=\bar{\phi}_{\bar{x}}(\bar{x},\bar{z}))$ of the problem defined by Eqs. (1,3) is dependent only on the shape function $z_T(x)$ and the parameter

$$(\gamma+1)k(M_\infty)\tau/\beta^3$$

If the shape function is the same in two cases and if the parameter $[(\gamma+1)k(M)_\infty)\tau/\beta^3]$ has the same value in both cases then the scaled velocity $\bar{u}(\bar{x},\bar{z})$ is the same. Since $(\gamma+1)$ is a constant $[(\gamma+1)=2.4]$, the similarity condition for $\bar{u}(\bar{x},\bar{z})$ is that

$$k(M_\infty)\tau/\beta^3 = \text{constant} \quad (4)$$

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*Senior Research Fellow, Dept. of Aeronautical Engineering.

or

$$\beta^2/[k(M_\infty)\tau]^{2/3} = K = \text{constant} \tag{5}$$

The second of these is the usual form of the transonic similarity parameter (K).

The pressure coefficient C_p is given to a first approximation, by

$$C_p = -2u(x,z) = -2\bar{u}(\bar{x},\bar{z})/\left\{\frac{(\gamma+1)k(M_\infty)}{\beta^2}\right\} \tag{6}$$

For a given value of similarity parameter K and a thickness/chord ratio τ then provided the functional dependence of $k(M_\infty)$ on M_∞ is known, the appropriate freestream Mach number can be calculated. In the examples discussed in Ref. 1 the form of $k(M_\infty)$ used is

$$k(M_\infty) = M_\infty^{3/2} \tag{7}$$

and $\tau = 0.06$.

In the comparison in Ref. 1 with the results of Murman for the value of $K = 1.8$ a slight error was made in the program input, and the Mach number used was $M_\infty = 0.87$ and

$$K = 1.8248$$

rather than the correct $K = 1.8$. If the correct M_∞ had been used in Ref. 1 then the shock wave would be 1% chord further aft than the predicted value, that is closer to Murman's[2] result; otherwise the changes in the C_p distributions are almost imperceptible.

In the calculation of Nørstrud's[3] result the particular form of $k(M_\infty)$ used is

$$k(M_\infty) = M_\infty^2 \tag{8}$$

For $\tau = 0.084$, $M_\infty = 0.8483$ the parameter K defined by Eq. (5) is then

$$K = 1.8205.$$

Thus the similarity parameters used both by Nixon[1] and by Nørstrud[3] are almost identical and both are very close to the value of $K (= 1.8)$ used by Murman[2]. To make a direct comparison the relation for $k(M_\infty)$ used by both Nixon[1] and Murman[2] is used, namely Eq. (7), together with $\tau = 0.06$. Thus for $K = 1.8205$ Nørstrud's[3] results can be taken to be equivalent to a freestream Mach number of $M_\infty = 0.8702$ compared with Nørstrud's derivation of $M_\infty = 0.8754$ that occurs if Nørstud's form for $k(M_\infty)$, Eq. (8), is used for the scaling. However, the use of Eq. (8) would make any comparison invalid.

Since $\bar{u}(\bar{x},\bar{z})$ is invariant with K the shock location will depend only on the value of K and hence the shock location in Nørstrud's[3] example should lie somewhere between the predicted locations of Nixon and Murman. Since $\bar{u}(\bar{x},\bar{z})$ is invariant with K the C_p distribution presented in Ref. (1) was obtained from Nørstrud's example by the transformation derived from Eqs. (6,7), that is

$$C_{p1} = \left[\left(\frac{M_2{}^2}{1-M_2{}^2}\right) C_{p2} \left(\frac{M_1{}^{3/2}}{1-M_1{}^2}\right)\right] \tag{9}$$

where $M_1 = 0.87$, $M_2 = 0.8483$; C_{p_1} and C_{p_2} are the respective pressure coefficients.

The figures and pressure transformations quoted by Nørstrud appear to be due to the use of a similarity parameter generally used for correlating experimental results rather than a direct comparison of the differential equations and their boundary conditions.

The second example is that for $K = 2.94$ which for $k(M_\infty)$ given by Eq. (7) and $\tau = 0.06$ is equivalent to a flow at $M_\infty = 0.8$. This example was computed by the finite difference method of Murman and Cole[4] but does not appear in Ref. 4. This should have been made clear in the text. Nørstrud's comments on this example are thus not applicable. In any event the results of Ref 4 are for $K = 3.0$ and for $k(M_\infty)$ given by Eq. (7), $\tau = 0.06$ thus gives $M_\infty = 0.7962$ not $M_\infty = 0.8085$ as suggested by Nørstrud. For comparison Nørstrud shows the results of an approximate compressibility correction. In the example the flow is considerably subcritical ($C_{p/\text{critical}} = -0.435$) and most compressibility corrections, including the 'first approximation' given by Nixon and Hancock[6], will give satisfactory results. The example was only computed to illustrate the convergence of the extended integral equations method to an accurate solution in a simple test case.

Nørstrud's second point concerns the validity of approximating the velocity in the flowfield by a decay function, specifically exponential decay. The main case for the argument is based on a comparison of velocity variations using the decay function with the computed values by Magnus et al.[7] for the flow around a parabolic arc aerofoil in a closed channel.

Generally in integral equation methods the decay function chosen will have a scale parameter $r(\bar{x})$ which may be determined[6] by making the McLaurin expansion of the decay function in $\bar{z}$ identical to the McLaurin expansion of the velocity to at least first order in $\bar{z}$ so that the decay function is at least correct close to the aerofoil surface. For example, if

$$\bar{u}(\bar{x},\bar{z}) = \bar{u}(\bar{x},o) f\left(\frac{\bar{z}}{r(\bar{x})}\right) \text{and } f(o) = 1 \tag{10}$$

then

$$\bar{u}(\bar{x},\bar{z}) = \bar{u}(\bar{x},o)\left\{1 + \frac{\bar{z}}{r(\bar{x})} f'(0) + \ldots\ldots\right\} \tag{11}$$

The expansion for

$$\bar{u}(\bar{x},o) + \bar{z}\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)_{\bar{z}=0} + \ldots.. \tag{12}$$

From irrotationality

$$\frac{\partial \bar{u}}{\partial \bar{z}} = \frac{\partial \bar{w}}{\partial \bar{x}} \text{ and } \left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)_{\bar{z}=0}$$

can be obtained from the boundary condition Eq. (3). Comparing the series Eqs. (11,12) therefore and matching the first two terms gives

$$r(\bar{x}) = \frac{\bar{u}(\bar{x},o) f'(0)}{(\partial \bar{w}/\partial \bar{x})_{\bar{z}=0}} \tag{13}$$

which for exponential decay gives

$$r(\bar{x}) = \frac{\bar{u}(\bar{x},o)}{k/\beta_3^2 \tau z_T''(x)} \tag{14}$$

Nørstrud appears to approximate $\bar{u}(\bar{x},o)$ by $k/\beta^3\ (u_i(x,o))$ where $u_i(x,o)$ is the incompressible value, and if this assumption is made then

$$r(x) = [u_i(x,o)/\tau z_T''(x)] \tag{15}$$

This approximation to $r(x)$ does not then give correct behavior for $|z/r(x)| << 1$ although for a biconvex aerofoil the error may not be too great. Incidentally the scale factor

$r(x)$ does not give the 'lateral extent' of the compressibility effects but is simply a measure of the initial rate of decay of the velocity $\bar{u}(\bar{x},\bar{z})$ with $\bar{z}$. Nørstrud's approximation to $r(\bar{x})$, Eq. (15), is not too inaccurate then it is not surprising that the comparison with Ref. 7 is qualitatively correct for $|\bar{z}/r(\bar{x})| < 1$. Indeed it would be surprising if it were not. In any event it is important in any integral equation method that the velocity in the neighborhood of the aerofoil is *quantitatively* correct in order to ensure adequate results in all cases, not just for the simple case of biconvex aerofoils.

The reason that Mach number plots were not included in Ref. 5 is that no comparable finite difference results, that is, flow in a freestream, were available in the literature; the case computed by Magnus et al.[7] being for a flow in a closed channel.

The third point that Nørstrud makes is his disagreement with the analysis of Ref. 5. No reason is given. It should be noted however that it is pointed out in Ref. 5 that Nørstrud's[3] basic integral equation *can* be written in the *form* of the first equation of Ref. 5. It is not suggested that the two equations are identical. The main point of the argument in Ref. 5 is that if the tangency boundary conditions are satisfied on the plane $z = \pm 0$ rather than on the aerofoil surface, then in the transonic integral equation there are two coupled nonlinear integral equations for the symmetric and anti-symmetric components of the flow, not one equations as implied in Ref. 3. A similar pair of equations, although uncoupled, are an established feature of thin aerofoil incompressible theory. Any transcription errors as regard signs in Ref. 5 do not obscure this fundamental point.

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Comment on "Gardon Heat Gage Temperature Response"

Alexander H. Flax*

Institute for Defense Analyses, Arlington, Va.

KITA and Laganelli[1] have undertaken to show by numerical examples that the Gardon circular thin disk instrument for measuring heat transfer rate has properties more or less independent of varying edge temperature. The gage measurement of heat transfer rate in the steady state depends on the difference between the center temperature and the edge temperature. In the situation considered by Kita and Laganelli, the heat transfer rate was assumed constant but the edge temperature was varying. Further, in the numerical example presented in Ref. 1, the edge temperature variation with time was taken to be linear starting at time zero. For this edge temperature variation, the explicit analytical solution is available in the literature for an unheated cylindrical body and may be added by the principle of superposition to the solution for heat addition uniform over the disk imposed as a Heaviside step function beginning at time equals zero. The sum of these two solutions gives the complete analytical solution to the problem analyzed numerically in Ref. 1 and leads to somewhat different conclusions than those arrived at by the authors.

The solution[2] of the partial differential equation of heat conduction for a cylindrical body initially at constant temperature throughout (taken to be zero) and subjected to a temperature increasing as ϵt from $t=0$ gives for the temperature, T, in the body

$$T(r,t) = \epsilon\left[t - \frac{(a^2-r^2)}{4\kappa}\right] + \frac{2\epsilon}{a\kappa}\sum_{n=1}^{\infty}\frac{e^{-\kappa\alpha_n^2 t}J_0(r\alpha_n)}{\alpha_n^3 J_1(a\alpha_n)} \tag{1}$$

where r is the radial coordinate, a is the radius of the disk, κ is the thermal diffusivity of the material, $K/\rho c$,c is the specific heat, ρ is the density, and K is the thermal conductivity. J_n is the Bessel function of the first kind and of the nth order; the α_n's are the roots of the equation $J_0(a\alpha_n)=0$. Also, for the cylindrical body with edge temperature held at zero and subjected at $t=0$ to a step in heat transfer uniform over the interior, the solution to the heat conduction equation is[2]

$$T(r,t) = \frac{A_0(a^2-r^2)}{4K\delta} - \frac{2A_0}{aK\delta}\sum_{n=1}^{\infty}\frac{e^{-\kappa\alpha_n^2 t}J_0(r\alpha_n)}{\alpha_n^3 J_1(a\alpha_n)} \tag{2}$$

where A_0 is the rate of heat transfer to the surface of the disk, δ is the thickness of the disk, and other terms are as previously defined.

Adding these two solutions gives, by superposition, the solution to the Gardon gage problem with linear varying edge temperature as

$$T(r,t) - \epsilon t = \frac{(A_0-\rho c\delta\epsilon)(a^2-r^2)}{4K\delta} - \frac{2(A_0-\rho c\delta\epsilon)}{aK\delta}\sum_{n=1}^{\infty} e^{-\kappa\alpha_n^2 t}\frac{J_0(r\alpha_n)}{\alpha_n^3 J_1(a\alpha_n)} \tag{3}$$

Thus, through comparison of Eqs. (2) and (3), we arrive at the conclusion that the steady-state temperatures as well as the time variations of the temperature differences between the interior points of the disk and the edge with linear variation in edge temperature, ϵt, and with heat addition, A_0, are of the same functional form as for constant edge temperatures. However, the temperatures in the disk correspond to those which would result from an equivalent heat transfer rate to the surface equal to $A_0 - \rho c\delta\epsilon$. Therefore, contrary to the conclusion reached by Kita and Laganelli, the measured heat transfer rate must, in general, be corrected for varying edge temperature. It is of course possible in specific cases that $\rho c\delta\epsilon$ may be sufficiently small relative to A_0 so that the correction may be neglected.

The close relationship between the temperature solutions for varying edge temperature and for heat input over the disk is not coincidental. Carslaw and Jaeger (Ref. 2, p. 294) show that in the typical heat conduction problem in which temperatures are arbitrarily specified at $t=0$, the temperatures may be considered to have been generated by instantaneous sources of heat of amount ρcT per unit volume, where T is taken relative to some reference temperature which is arbitrarily taken to be zero. In a more general case, the heating may be considered continuous of amount A' per unit volume such that $A' = \rho c dT/dt$. Moreover, if the reference value of

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*President. Fellow AIAA.